Introduction to Game Theory and Congestion Games

Zhenyu Yang

May 6, 2025

Overview

Today, we'll try to deepen our understanding of the traffic assignment problem from the perspective of game theory

- (Re)familiarize with classical game theory concepts and discuss applications in transportation problems
- A recap on the concept of user equilibrium
- We will study a class of games in game theory closely related to transportation economics
- A quick skim of different variants of equilibria in transportation modeling

Introduction

- Game theory is one of the most important tools of economics
  - Players make a choice or multiple choices between actions
  - Their outcome (or pay-off) is not only based on their own decision, but also on that of other players
  - Players have preferences over outcomes
- Games differ in many aspects
  - Timing. When and how many times player make decisions?
  - Observations. Can players observe each other's choices?
  - Uncertainty. Do some players have information that the others do not?
- Main question: What is my optimal decision, given the decision of my opponents (or expectations)?

Examples

Cuban missile crisis in October 1962 [1,2] :

- USSR: maintaining (M) or withdrawing (W) their missiles
- USA: an air strike (A) or a simple naval blockade (B)
- (AM) would result in a nuclear war, (BW) in a compromise and the other two in a victory for either one of them

[1] Brams, Steven J. "Game theory and the Cuban missile crisis." Plus Magazine 1 (2001)

[2] http://users.humboldt.edu/ogayle/hist111/CubanMissileCrisis.html

Introduction

- Transportation is heavily influenced by interactions of individuals among each other, as well as between individuals and operators
  - Drivers, passengers, pedestrians
  - Ride-hailing platforms
  - Transportation management authorities
- Based on whether agents know others' decisions before making their decisions:
  - Simultaneous game
    - Prisoner's Dilemma
    - Rock-Paper-Scissors
  - Sequential game
    - Chess
    - Auction
- Main question: What is my optimal decision, given the decision of my opponents (or expectations)?

Simultaneous game

- We begin by looking into games with two players
  - Both agents make rational decisions
  - Both agents have to decide at the same time
  - They do not know what their opponent will choose, but they do know their own payoffs and the payoffs of their opponent in every scenario
- The most common way to represent (2-player) simultaneous move games is in a matrix form.
  - The "cell" that emerges is the outcome of the game.
  - Traditionally, the first entry in a cell represents the payoff of the row player, and the second entry is the payoff of the column player.

Simultaneous game

- Example: Ride-pooling
  - A driver offers a trip to two individuals for 5 CHF
  - If an individual is open to share her ride, she receives a discount of 1 CHF
  - If both individuals share, they perceive inconvenience from a detour, worth 2 CHF
  - Nash Equilibrium: A situation where no player can improve their outcome by changing their strategy alone, given the strategies of the others
  - Pure strategy Nash Equilibrium: No player randomizes his/her decision
  - Two pure strategy Nash Equilibria (NE)

	Share	Not Share
Share	(6, 6)	(4, 5)
Not Share	(5, 4)	(5, 5)

Table: Cost matrix for the ride-pooling game

Simultaneous game

- Example: Ride-pooling
  - What if the discount is 3 CHF?
    - This leads to a single NE: (Share, Share)

	Share	Not Share
Share	(4, 4)	(2, 5)
Not Share	(5, 2)	(5, 5)

  - What if the discounts are 2 CHF for A and 3 CHF for B if both share, 1 CHF for A and 0 CHF for B if either A or B does not share, and 0 CHF for A and 1 CHF for B if neither A nor B shares?

	Share	Not Share
Share	(5, 4)	(4, 5)
Not Share	(4, 5)	(5, 4)

    - This leads to no PSNE

Simultaneous game

- Mixed Strategy Nash Equilibrium: At least one player randomizes his/her decision

	Share (p)	Not Share (1- p)
Share (p)	(5, 4)	(4, 5)
Not Share (1 - p)	(4, 5)	(5, 4)

  - Player A is indifferent (and therefore willing to randomize) if player B shares with probability 0.5
  - Note that the game is symmetric, so the same reasoning applies to player B
  - Mixed Strategy Nash Equilibrium: ((0.5,0.5), (0.5,0.5))

Duopoly competition

- Classical competition model (19th century) where firms choose quantities/prices, and prices/quantities are determined through market movements (introduced by Prof de Palma in the first lecture)
  - Cournot competition: both companies choose their quantities simultaneously
  - Bertrand competition: both companies choose their prices simultaneously
- Example: Competition of Uber (1) and Lyft (2)
  - Demand for ride-hailing vehicles has the following inverse demand function: $P = 5 - 0.01(q_1 + q_2)$
    - P is the price of ride-hailing trips
    - q_1 and q_2 are the total numbers of vehicles deployed by Uber and Lyft, respectively
  - Both firms have marginal costs of deploying vehicles equal to 3 per demand unit
  - Question: What is the equilibrium number of vehicles Uber and Lyft will dispatch?

Duopoly competition

- Cournot: choosing quantities simultaneously
  - Splitting the inverse demand function for the two firms gives:

$$P = 5 - 0.01(q_1 + q_2)$$

  - The revenue for firm one is then as follows:

$$pq_1 = 5q_1 - 0.01q_1^2 - 0.01q_1q_2$$

  - At optimality, MR = MC, yielding

$$3 = 5 - 0.02q_1 - 0.01q_2$$

  - Here we note symmetry in the cost/profit functions of the two firms, which allows to solve for both variables

$$3 = 5 - 0.02q_1 - 0.01q_2$$

  - $q_1 = q_2 = 66.66, P = 3.66$

Duopoly competition

- Bertrand: choosing prices simultaneously
  - A firm has no profit if their price is higher than the price of its competitor, therefore a firm will always decrease its price
  - This can continue until the price is equal to the marginal cost, so:

$$p_1 = p_2 = 3$$

  - $Q = 200$ such that $q_1 = q_2 = 100$
- Which one is more realistic?
  - If capacity and output are difficult to adjust $->$ Cournot
  - Example: Flights in busy seasons are set far in advance, when they determine their quantity (=plane capacity * flights)
  - If capacity and output can be changed easily $->$ Bertrand
  - Example: price of bottled water in supermarkets

Sequential game

- In some situations, a simultaneous game where all agents decide at the same time may not be realistic
  - For many interactions between an operator and a customer, their decisions are made sequentially
  - Competition between competitors can also be sequential
- Example: New ride-hailing competition
  - In a city, Uber has been the sole ride-hailing company up to now.
  - Lyft is thinking about joining the market in this city
  - As a response, Uber can lower their fares to enhance competition
  - In this example, Uber first decides, and Lyft responds thereafter

Sequential game

- Stackelberg competition: Player 1 chooses first, Player 2 observes this decision and makes a decision afterwards
- Example: Uber has more power and decides first, and Lyft follows
  - Recall: $P = 5 - 0.01(q_1 + q_2)$
  - For Lyft, the optimal quantity is $q_2 = 100 - 0.5q_1$, see Cournot for derivation of their Best response!
  - Insert this in the revenue function of Uber:
  - $pq_1 = 5q_1 - 0.01q_1^2 - 0.01q_1q_2 = 4q_1 - 0.005q_1^2$
  - At optimality, MR = MC, yielding: $3 = 4 – 0.01q_1$
  - This gives $q_1 = 100$ and $q_2 = 50$
  - Price = 3.5
  - Profit Uber: 50, Profit Lyft: 25
  - In Cournot, Profit Uber: 44, Profit Lyft: 44

Games with many players

So far, we have introduced some typical games in transportation

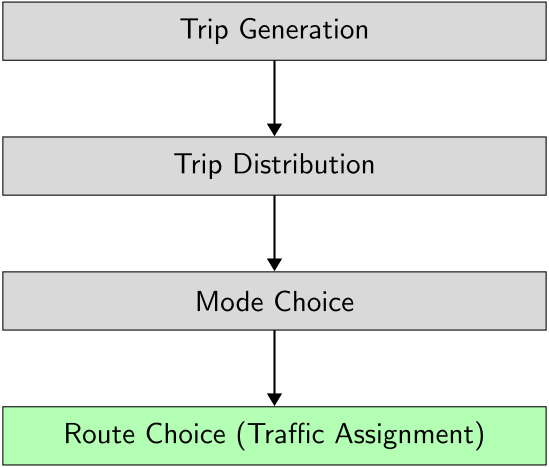
- Two or multiple players
- Companies, drivers, passengers ....

In practice, the number of players can be large.

- A typical example is transportation planning

Urban transportation planning

Urban transportation planning is traditionally carried out using a four-step method:



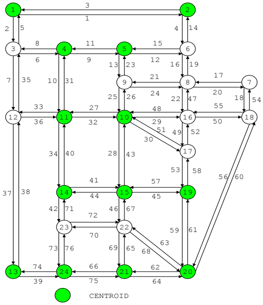
Urban transportation planning

1. Trip generation: the total number of trips made to and from each zone
2. Trip distribution: the total number of trips between origin-destination (OD) pairs
3. Mode choices. the number of trips will be made by car, bus, bike, etc., between each OD pair
4. Route choice, or traffic assignment, involves assigning travelers to different paths

- Originally developed in the 1950s and 1960s when planning major highway facilities
  - This model is not without its limitations, and alternative paradigms have been suggested. Nevertheless, it remains common in practice.
- In this lecture, we focus on the fourth step

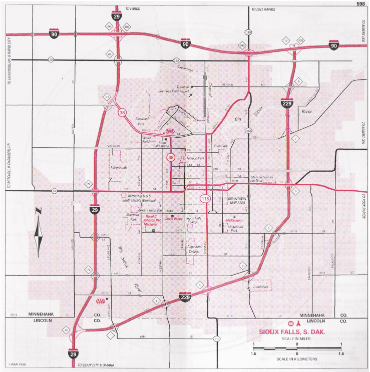
Road network representation

Consider a directed network $G = (V, E)$, where V and E are the set of nodes and links, respectively.



Sioux Falls Test Network

Prepared by Hai Yang and Meng Qiang, Hong Kong University of Science and Technology

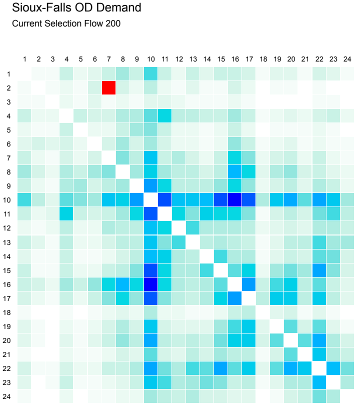


Adopted from the GitHub repository

Travel demand

Let $Z \in V^2$ denote the set of OD pairs

- The demand between an OD pair $z \in Z$ is represented by d_z
- Example: the red dot is the demand from node 2 to node 7



Shared by the GitHub repository

Path flow

Let P_z denote the set of paths between OD pair $z \in Z$.

- Each path p is a sequence of links $e \in E$
- Set $P = \cup_{z \in Z} P_z$

Let f_p denote the flow on path $p \in P$.

- The number of travelers using path p

Let f denote the vector of all path flows f_p.

- f is feasible iff the following conditions are satisfied:
  1. $f_p \geq 0$, for all $p \in P$
  2. $\sum_{p \in P_Z} f_p = d_z$ for all $z \in Z$
- Paths used in transportation networks are usually acyclic
  - A network is acyclic if there are no cyclic paths in the network.
- We use the words 'route' and 'path' interchangeably

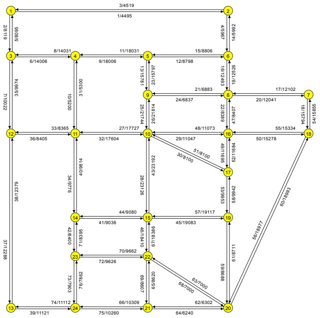
Link flow

Link flows can be determined from path flows.

- The traffic flow on link $e \in E$ is

$$x_e = \sum_{p \in P} \delta_e^p f_p$$

$$\delta_e^p = \begin{cases} 0 & \text{if link } e \text{ is not part of } p, \\ 1 & \text{otherwise} \end{cases}$$

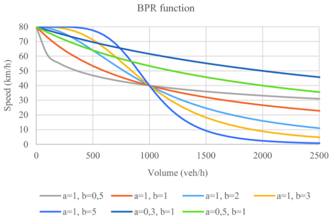


Congestion

Each link $e \in E$ has a travel time function $t_e(x_e)$

The travel time function captures congestion effects.

- $t_e(x_e)$ is non-negative and non-decreasing
- The functions t_e varies with links e
- A widely used one: Bureau of Public Roads (BPR) function
  $t_e(x_e) = t_e^0 \left(1 + \alpha \left(\frac{x_e}{C_e}\right)^\beta\right)$, where C_e is the link capacity



Adopted from Saric, A., Albinovic, S., Dzebo, S., Pozder, M. (2019)

User equilibrium

A path flow pattern f is a user equilibrium (UE) iff,

- for any path $p \in P$ with $f_p > 0$, there does not exist a path $p' \in P$ such that

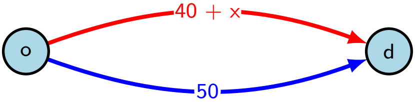
$$\sum_{e \in p'} t_e(x_e) < \sum_{e \in p} t_e(x_e).$$

  - No user can reduce the individual travel time by unilaterally changing his/her route choices
  - For each OD pair, all used paths between them have equal and minimal travel time.
- This is simply the Nash equilibrium of the large-population game where no individual's route choice affects overall traffic.
- Proposed by John Glen Wardrop in 1952 and therefore often termed as Wardrop equilibrium

Example of two paths

We can use the definition to solve for equilibrium on simple networks.

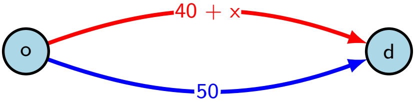
- If there are 30 vehicles choosing these paths, how many choose the red and the blue paths, respectively?



Example of two paths

We can use the definition to solve for equilibrium on simple networks.

- If there are 30 vehicles choosing these paths, how many choose the red and the blue paths, respectively?



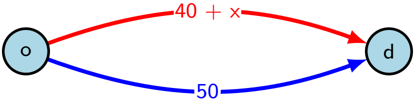
  - 10 choose the red path, and the rest choose the blue path
  - Both paths have a travel time of 50 minutes.

- What if there are only 9 vehicles choosing these paths?

Example of two paths

We can use the definition to solve for equilibrium on simple networks.

- If there are 30 vehicles choosing these paths, how many choose the red and the blue paths, respectively?



  - 10 choose the red path, and the rest choose the blue path
  - Both paths have a travel time of 50 minutes.

- What if there are only 9 vehicles choosing these paths?
  - All drivers choose the red path
  - The blue path has a higher travel time

System optimum

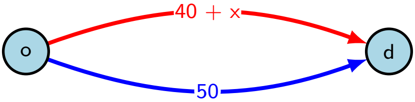
Two possible traffic assignment rules:

- User equilibrium (UE): A feasible assignment in which all used paths have equal and minimal travel times.
- System optimum (SO): A feasible assignment that minimizes the total travel time

$$
\begin{aligned}
\min \quad & \sum_{e \in E} t_e(x_e) x_e & \\
\text{s.t.} \quad & x_e - \sum_{p \in P} \delta_{ep} f_p = 0 & \forall e \in E \\
& \sum_{p \in P_z} f_p = d_z & \forall z \in Z \\
& f_p \geq 0 & \forall p \in P
\end{aligned}
$$

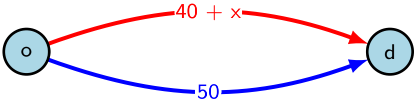
SO in the example of two paths

- If there are 30 vehicles choosing these paths and the total travel time is minimized, how many choose the red and the blue paths, respectively?



SO in the example of two paths

- If there are 30 vehicles choosing these paths and the total travel time is minimized, how many choose the red and the blue paths, respectively?



  - 5 choose the red path, and the rest choose the blue path
  - The red path has a travel time of 45 minutes, while the blue path has a travel time of 50 minutes

Price of anarchy (POA): the ratio between the total travel times at UE and SO, respectively

- The total travel time at SO is $45 \times 5 + 50 \times 25 = 1475$
- The total travel time at UE is $50 \times 30 = 1500$
- $\text{POA} = 1500/1474$

The "invisible hand" functions well in this case.

How to find UE

At this point, there are three important questions you might be asking at this point:

- Does a user equilibrium solution always exist?
- If so, is the user equilibrium solution unique?
- Is there any practical way to find an equilibrium in general networks?

Congestion Games

A congestion game is defined by a tuple (J, E, P, t), where:

1. A set of players (drivers) J
2. A set of facilities (links) E
3. For each player j, a set of actions (paths) P_j. Each action (path) $p_j \in P_j$ represents a subset of the facilities (links) : $p_j \subseteq E$.
4. For each facility (link) $e \in E$, a cost function $t_e : \mathbb{N} \to \mathbb{R}_{\geq 0}$.

Player costs are then defined as follows.

- For action profile $p = (p_j)_{j \in J}$, define $x_e(p) = |\{j : e \in p_j\}|$ to be the number of players using facility (link) e. Then the cost (travel time) of player (driver) j is:

$$c_j(p) = \sum_{e \in p_j} t_e(x_e(p))$$

Best response dynamics

Consider the following iterations:

1. Players (drivers) start playing arbitrary actions (paths).
2. In arbitrary order, players take turns changing their actions if doing so can lower their individual cost.
3. Stop until no one can lower their individual cost ...

If best response dynamics stops, it admits a pure strategy Nash equilibrium.

But will the iteration process ever stop?

- For example, the game of Rock Paper Scissors never stops.

The answer is yes.

- Best response dynamics (BRD) always halt in congestion games.

Why?

Best response dynamics

- By definition, in each round of BRD, if player j switches from p_j to $q_j \in P_j$, play j's cost must be reduced, i.e.,

$$\begin{aligned} & c_j(q_j, p_{-j}) - c_j(p_j, p_{-j}) \\ = & \sum_{e \in q_j \setminus p_j} t_e(x_e(p) + 1) - \sum_{e \in p_j \setminus q_j} t_e(x_e(p)) < 0 \end{aligned}$$

- Consider the function $\phi : P \to \mathbb{R}$:

$$\phi(p) = \sum_{e \in E} \sum_{k=1}^{x_e(p)} t_e(k)$$

  We call the function $\phi(\cdot)$ as a **potential function** (Note: *$\phi(p)$ is not the total cost*)

How does ϕ change in each round of BRD?

Potential game

The change of ϕ is the same as that in play (driver) j's cost !

$$\phi(p) = \sum_{e \in E} \sum_{k=1}^{x_e(p)} t_e(k)$$

- If player j switches from action (path) p_j to $q_j \in P_j$,

$$\begin{aligned} & \phi(q_j, p_{-j}) - \phi(p_j, p_{-j}) \\ = & \sum_{e \in q_j \setminus p_j} t_j(x_e(p) + 1) - \sum_{e \in p_j \setminus q_j} t_e(x_e(p)) \\ = & c_j(q_j, q_{-j}) - c_j(p_j, p_{-j}) \end{aligned}$$

- The value of ϕ will get smaller by iterations
- ϕ can take on only finitely many values, and therefore BRD will stop eventually

Thus, we assert that the best response dynamics always halt in congestion games

Potential game

Formally, a function $\phi : P \to \mathbb{R}$ is a potential function if for all $j \in J$, $p \in P$, and $q_j \in P_j$:

$$\phi(q_j, p_{-j}) - \phi(p_j, p_{-j}) = c_j(q_j, q_{-j}) - c_j(p_j, p_{-j})$$

- A game is a potential game if it admits a potential function

Theorem

Every finite potential game has a pure strategy Nash Equilibrium (PSNE)

Proof:

- Consider a action profiles p under which ϕ is minimal.
- By definition, no player can benefit by deviating from using a pure strategy.
- Hence, p must be a PSNE.

Congestion game and traffic assignment

But if we utilize the BRD iteration, it takes exponential time to find the equilibrium.

Congestion game and traffic assignment

But if we utilize the BRD iteration, it takes exponential time to find the equilibrium.

- Not practical in traffic assignments with a large population

Congestion game and traffic assignment

But if we utilize the BRD iteration, it takes exponential time to find the equilibrium.

- Not practical in traffic assignments with a large population

Then, how do we conduct traffic assignment in practice?

Non-atomic congestion game

Now, we consider a continuum of drivers that are infinitesimally small. Still, we have

- The set of congestible facilities (links) E remains the same.
- For each facility (link) $e \in E$, a cost function $t_e : \mathbb{N} \to \mathbb{R}_{\geq 0}$.

Now we consider the set of types of players Z

- Plays of the same type share the same origin and destination
- The amount of players of type z is d_z.
- Each type z of players to distribute fractionally over their action(path) set P_z
  - Let f_p represent the amount of players using action p.

$$\sum_{p \in P_Z} f_p = d_z$$

Recall the user equilibrium model ...

Potential function

A path flow f is a Nash Equilibrium iff for any action (path) $p \in P$ with $f_p > 0$, there does not exist a path $p' \in P$ such that

$$\sum_{e \in p'} t_e(x_e) < \sum_{e \in p} t_e(x_e).$$

The non-atomic analogue of the potential function from atomic games:

$$\phi(p) = \sum_{e \in E} \sum_{k=1}^{x_e(p)} t_e(k)$$

$$\Updownarrow$$

$$\phi(f) = \sum_{e \in E} \int_0^{x_e(f)} t_e(y)dy$$

- A player's strategy choice does not affect the delays

Similar to the atomic case, a minima of this function is a Nash Equilibrium.

Beckmann formulation

Using the potential function of the non-atomic game, the equilibrium solves the following convex optimization program:

$$
\begin{aligned}
\min \quad & \sum_{e \in E} \int_0^{x_e} t_e(y) dy & \\
\text{s.t.} \quad & x_e - \sum_{p \in P} \delta_{ep} f_p = 0 & \forall e \in E \\
& \sum_{p \in P_z} f_p = d_z & \forall z \in Z \\
& f_p \geq 0 & \forall p \in P
\end{aligned}
$$

Optimality conditions of convex optimization

Consider the following general optimization problem.

$$\min_{x \in \mathbb{R}^n} f(x)$$
$$\text{s.t. } h_i(x) \leq 0, \quad i = 1, \dots m$$
$$g_j(x) = 0, \quad j = 1, \dots r$$

The Karush-Kuhn-Tucker conditions or KKT conditions are:

- $0 = \nabla f(x) + \sum_{i=1}^{m} u_i \nabla h_i(x) + \sum_{j=1}^{r} v_j \nabla g_j(x)$
- $u_i \cdot h_i(x) = 0$ for all i
- $h_i(x) \leq 0, g_j(x) = 0$ for all i, j
- $u_i \geq 0$ for all i

KKT conditions are sufficient and necessary for the optimality of convex optimization programs

Optimality conditions

By writing the KKT, the optimality condition of the Beckmann formulation is given by

$$
\begin{aligned}
f_p &\geq 0 && \forall p \in P \\
c_p - \kappa_z &\geq 0 && \forall z \in Z \\
f_p(c_p - \kappa_z) &= 0 && \forall p \in P \\
\sum_{p \in P_z} f_p &= d_z && \forall z \in Z
\end{aligned}
$$

- The second condition shows that κ_z is the *shortest path travel time* for OD pair z
- The third condition shows that if a path is used ($f_p > 0$) its travel time must be equal to κ_z
- The last condition is the flow conservation condition

These are exactly the definitions of user equilibrium

Uniqueness of link flow

Claim: The equilibrium solution is unique in link flows $x = (x_e)_{e \in E}$.

- The potential function is strictly convex in the link flows x if the link performance functions are increasing.
- The convexity of the potential function regarding link flows is shown by writing the Hessian
  - Recall
$$\phi = \sum_{e \in E} \int_0^{x_e} t_e(y)dy$$
  - The first derivative $\partial \phi / \partial x_e = t_e(x_e)$
  - The second partial derivative $\frac{\partial^2 \phi}{\partial x_e \partial x_{e'}} = t_e'(x_e)$, if $e = e'$,, Otherwise, $\frac{\partial^2 \phi}{\partial x_e \partial x_{e'}} = 0$.
  - What does the Hessian of ϕ look like?

Uniqueness of link flow

Claim: The equilibrium solution is unique in link flows $x = (x_e)_{e \in E}$.

- The potential function is strictly convex in the link flows x if the link performance functions are increasing.
- The convexity of the potential function regarding link flows is shown by writing the Hessian
  - Recall
$$\phi = \sum_{e \in E} \int_0^{x_e} t_e(y) dy$$
  - The first derivative $\partial \phi / \partial x_e = t_e(x_e)$
  - The second partial derivative $\frac{\partial^2 \phi}{\partial x_e \partial x_{e'}} = t'_e(x_e)$, if $e = e'$,, Otherwise, $\frac{\partial^2 \phi}{\partial x_e \partial x_{e'}} = 0$.
  - What does the Hessian of ϕ look like?

Algorithm

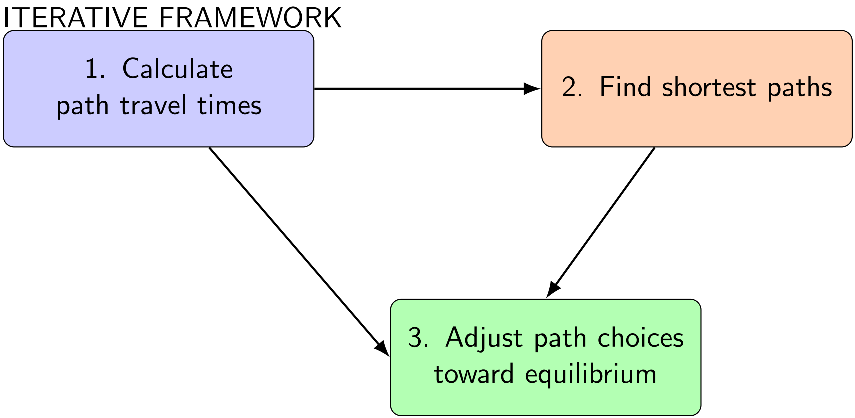
- The Hessian is a diagonal matrix, and its diagonal entries are strictly positive if $t'_e(x_e) > 0$.

The Hessian Matrix looks like:

$$\nabla^2 z(x) = \begin{bmatrix} \frac{\partial t_1(x_1)}{\partial x_1} & 0 & 0 & \cdots \\ 0 & \frac{\partial t_1(x_2)}{\partial x_2} & 0 & \cdots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \frac{\partial t_A(x_A)}{\partial x_A} \end{bmatrix} \tag{1}$$

Thus, the solution regarding link flows is unique.

Algorithm


ITERATIVE FRAMEWORK
1. Calculate path travel times
2. Find shortest paths
3. Adjust path choices toward equilibrium

Algorithm

So, this suggests one specific implementation of the iterative framework:

1. Start with some feasible link flow solution x.
2. Calculate the link travel times using the flows x.
3. Find the shortest paths between all origins and destinations.
4. Find the all-or-nothing link flows x^* corresponding to these shortest paths.
5. Choose $\lambda \in [0, 1]$ and update $x \leftarrow \lambda x^* + (1 - \lambda)x$
6. If "close enough to equilibrium" stop, otherwise return to step 2.

Variants of user equilibrium

So far, we have found a way to characterize the standard user equilibrium with Beckmann's formulation.

There are many variants of UE that occurred during the past decades:

- Elastic demand
- Link interactions
- Multiple classes of users ...

Can we always find a potential function?

Link interactions

In practice, a link's travel time may also be affected by other links' flow

- Highways where overtaking is allowed
- Ramp metering
- Spillback

Travel time function is no longer $t_e(x_e)$ but $t_e(x)$

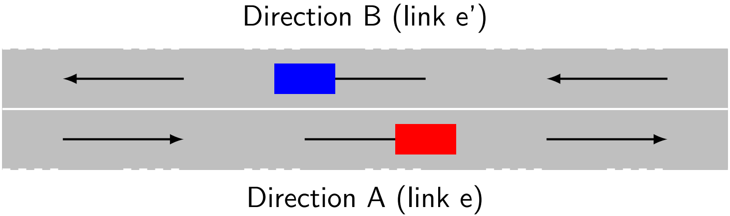
A natural question: can we just revise the Beckmann function in some way to capture such interaction?

$$\Phi = \sum_{e \in E} \int_0^{x_e} t_e(y) dy$$

Symmetric two-way link interactions

The simplest case of link interaction

- Pairwise relationships between the two (opposite direction) links representing two-way streets.



- Consider link e and anther link $e' \in E$ in the opposite direction
  - The travel times on links e and e' are given by

$$t_e = t_e(x_e, x_{e'}), t_{e'} = t_{e'}(x_{e'}, x_e)$$

  - Symmetric interaction

$$\frac{\partial t_e}{\partial x_{e'}} = \frac{\partial t_{e'}}{\partial x_e}$$

Symmetric two-way link interactions

$$\phi = \frac{1}{2} \sum_{e \in E} \int_0^{x_e} \left[t_e(y, x_{e'}) + t_e(y, 0) \right] dy$$

Two terms in the integral:

- First term: the flow in the opposite direction is held constant.
- Second term: the flow on the opposite link is held at zero.

We can verify that,

$$\frac{\partial \phi}{\partial x_e} = t_e(x_e, x_{e'})$$

Thus, we still have the optimality conditions from KKT:

$$\begin{aligned}
f_p &\geq 0 & \forall p \in P \\
c_p - \kappa_z &\geq 0 & \forall z \in Z \\
f_p(c_p - \kappa_z) &= 0 & \forall p \in P \\
\sum_{p \in P_z} f_p &= d_z & \forall z \in Z
\end{aligned}$$

Toll design

- Understand the toll design problem as a Stakelberg game
  - Two-level decision process:
    - **Leader:** Sets tolls on links (e.g., government or system planner).
    - **Followers:** Travelers choose routes minimizing personal cost (time + toll).
  - Perfect information assumption: Followers observe leader's decision.
  - Goal: Design tolls to induce socially optimal or user equilibrium flow.
- Marginal cost pricing: If we charge each traveler a toll of $\frac{\partial t_e(x_e)}{\partial x_e}$ when they pass link e
  - the generalized link cost of link e is therefore

$$\hat{t}_e(x_e) = t_e(x_e) + x_e \frac{\partial t_e(x_e)}{\partial x_e}$$

Marginal Cost Pricing and Beckmann's Formulation

- We can replace $t_e(x_e)$ with $\hat{t}_e(x_e)$ in the Beckmann formulation:

$$
\begin{aligned}
\min \sum_{e \in E} \int_0^{x_e} \hat{t}_e(y)\, dy \\
\text{s.t. } x_e - \sum_{p \in P} \delta_{ep} f_p = 0 \quad \forall e \in E \\
\sum_{p \in P_z} f_p = d_z \quad \forall z \in Z \\
f_p \geq 0 \quad \forall p \in P
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
\min \sum_{e \in E} t_e(x_e) x_e \\
\text{s.t. } x_e - \sum_{p \in P} \delta_{ep} f_p = 0 \quad \forall e \in E \\
\sum_{p \in P_z} f_p = d_z \quad \forall z \in Z \\
f_p \geq 0 \quad \forall p \in P
\end{aligned}
$$

- Under the marginal cost pricing scheme, travelers make route choices in a **system optimal** manner.
- Is the magical cost toll the only toll scheme that can achieve system optimum?

Thanks!

Zhenyu Yang

Urban Transport Systems Laboratory (LUTS)

École polytechnique fédérale de Lausanne (EPFL)

zhenyu.yang@epfl.ch

References

1. Boyles, S.D., Lownes, N.E. and Unnikrishnan, A., 2020. Transportation network analysis. Volume I, Version 0.85.
2. Sheffi, Y., 1985. Urban transportation networks (Vol. 6). Prentice-Hall, Englewood Cliffs, NJ.
3. Transportation Networks for Research Core Team. Transportation Networks for Research. https://github.com/bstabler/TransportationNetworks. 04, 29, 2024.